(Strict) coherence on Łukasiewicz events: geometry, probability and logic

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A glimpse on (strict) coherence

In 1931 Bruno de Finetti provided a general justification for the probabilistic representation of rational degrees of belief defining the *probability* of an unknown event a as the *fair price* that a rational Gambler is willing to pay to participate in a betting game, against the Bookmaker, the payoffs of which are 1 in case a occurs, and 0 otherwise.

Based on this very simple idea, de Finetti showed that all theorems of Kolmogorov's probability theory may be derived as consequences of his *coherence* criterion on assignments (books) on logically connected events.

Consider a finite class of (unknown), classical (yes/no), events $\Phi = \{e_1, \ldots, e_k\}$ and two players: the *Bookmaker* $\mathcal{B}$ and the Gambler $\mathcal{G}$.

- $\mathcal{B}$ publishes her betting quotients in a *book*, i.e., a map $\beta : \Phi \to [0, 1]$.
- $\mathcal{G}$ decides stakes σ_i (positive or negative!) and pays $\sum_{i=1}^{k} \sigma_i \cdot \beta(e_i)$.
- Once a truth-valuation h is realized, $\mathcal{B}$ pays back to $\mathcal{G}$ the amount $\sum_{i=1}^{k} \sigma_i \cdot h(e_i)$.
- The *balance* for the bookmaker *in h* is $\sum_{i=1}^{k} \sigma_i(\beta(e_i) - h(e_i))$.

Given Φ, a book β is said to be:

- *Coherent* if for every choice of stakes $\sigma_1, \ldots, \sigma_k$, there exists at least a possible world h in which Bookmaker's balance is not negative, i.e.,

$$\sum_{i=1}^{k} \sigma_i(\beta(e_i) - h(e_i)) \geq 0$$

- *Strictly-coherent* is for every choice of stakes $\sigma_1, \ldots, \sigma_k$, if there exists a possible world h in which

$$\sum_{i=1}^{k} \sigma_i(\beta(e_i) - h(e_i)) < 0$$

  then, there must exists another possible world h' in which

$$\sum_{i=1}^{k} \sigma_i(\beta(e_i) - h'(e_i)) > 0.$$

MV-algebras are structures in the signature $\langle \oplus, \neg, 0 \rangle$ of type $(2, 1, 0)$ which forms the variety generated by the standard algebra:

$$[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$$

where

$$a \oplus b = \min\{1, a+b\}, \neg a = 1 - a, 0 = 0.$$

The variety $\mathbb{MV}$ of MV-algebras is the *equivalent algebraic semantics* of Łukasiewicz infinite-valued logic, Ł$_\infty$. Thus, in particular, formulas of Ł$_\infty$ are naturally translated into algebraic terms in the language of MV-algebras so that *tautologies* naturally corresponds to *theorem*.

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Further operations are definable as follows

$$\begin{aligned}
&a \odot b = \neg(\neg a \oplus \neg b) = \max\{0, a+b-1\};\\
&a \to b = \neg a \oplus b = \min\{1, 1-a+b\};\\
&1 = \neg 0;\\
&a \wedge b = a \odot (a \to b) = \min\{a, b\};\\
&a \vee b = \neg(\neg a \wedge \neg b) = \max\{a, b\}.
\end{aligned}$$

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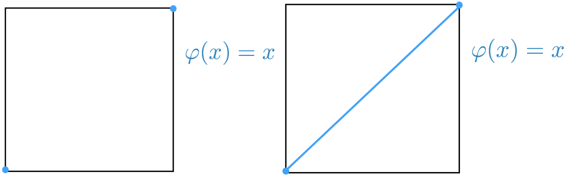
(3). The free MV-algebra over n free generators $\mathcal{F}_n$. This is the MV-subalgebra of $[0,1]^{[0,1]^n}$ of those functions which are (i) continuous; (ii) piecewise linear; (iii) each piece has integer coefficient. These are usually called *McNaughton functions*.



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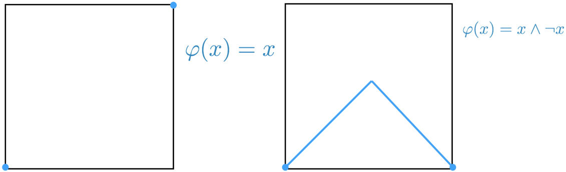

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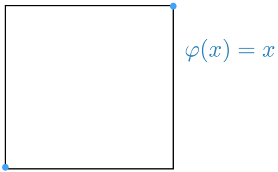

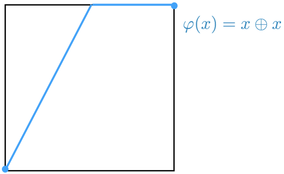

(1). Every boolean algebra is an MV-algebra in which $a \vee \neg a = 1$ holds. In other words, the variety of boolean algebras is a proper subvariety of MV-algebras. (Therefore, we shall henceforth only deal with MV-algebras with any loss of generality)

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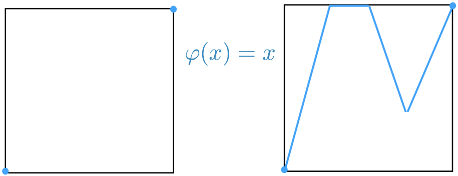

Fix any finite subset $\Phi = \{f_1, \ldots, f_k\}$ of $\mathcal{F}_n$, the free MV-algebra over n-free generators and let $\beta : \Phi \to [0,1]$ be a book. Then:

1. β is *coherent* if for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ there exists an MV-homomorphism $h : \mathcal{F}_n \to [0,1]_{MV}$ (i.e., a *possible world*) such that

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2. β is *strictly-coherent* if for all $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, if there exists an MV-homomorphism $h : \mathcal{F}_n \to [0,1]_{MV}$ (i.e., a *possible world*) such that

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then, there must exists another homomorphism h' such that

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The probability of (strict) coherence

Theorem

Fix a finite subset of classical formulas (i.e., classical events) $\Phi = \{e_1, \ldots, e_k\}$ and a book $\beta : \Phi \to [0,1]$. Then

1. β is coherent iff it extends to a finitely additive probability measure on the boolean algebra spanned by events in Φ. ([a])
2. β is strictly-coherent iff it extends to a regular and finitely additive probability measure on the boolean algebra spanned by events in Φ. ([b] and [c])

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[a] B. de Finetti, Theory of Probability, vol. 1, John Wiley and Sons, New York, 1974.

[b] J.G. Kemeny, Fair bets and inductive probabilities, JSL 20(3): 263–273, 1955.

[c] T. Flaminio, H. Hosni, F. Montagna, Strict coherence on many-valued events, JSL 83(1): 55–69, 2018.

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As for MV-algebras...

Definition (Mundici, 1995)

By a *state* of an MV-algebra $\mathbf{A}$ we mean a function $s : A \to [0, 1]$ such that $s(1) = 1$ and, $s(a \oplus b) = s(a) + s(b)$, whenever $a \odot b = 0$.

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Proposition (Kroupa, 2005 and Panti, 2009)

A map $s : \mathcal{F}_n \to [0,1]$ is a state iff s is of the form

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu$$

for μ being a Borel regular probability measure on $[0,1]^n$.

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Theorem (Mundici, 2006)

A book β on a finite subset Φ of a free MV-algebra $\mathcal{F}_n$ is coherent iff there exists a state s of $\mathcal{F}_n$ which extends β.

Definition (Mundici, 1995)

A *faithful state* of an MV-algebra **A** is state $s : A \to [0, 1]$ such that $s(a) = 0$, implies $a = 0$.

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Proposition (Flaminio, 2018)

A map $s : \mathcal{F}_n \to [0,1]$ is a faithful-state iff s is of the form

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Theorem (Flaminio, 2018)

A book β on a finite subset Φ of a free MV-algebra $\mathcal{F}_n$ is strictly-coherent iff there exists a faithful state s of $\mathcal{F}_n$ which extends β.

The geometry of coherence

For every finite subset $\Phi = \{f_1, \ldots, f_k\}$ of a free MV-algebra $\mathcal{F}_n$, we shall denote by

$$\mathscr{D}_\Phi = \{\beta : \Phi \to [0,1] \mid \beta \text{ is coherent}\}$$

and

$$\mathscr{S}_\Phi = \{\beta : \Phi \to [0,1] \mid \beta \text{ is strictly coherent}\}$$

Thus, a book $\beta : \Phi \to [0,1]$, displayed as $\beta = \langle \beta(f_1), \ldots, \beta(f_k) \rangle$ is a point of $[0,1]^k$. Therefore, $\mathscr{D}_\Phi$ and $\mathscr{S}_\Phi$ are subsets of $[0,1]^k$ as well.

For each subset $\Phi = \{f_1, \ldots, f_k\}$ of $\mathcal{F}_n$, both $\mathscr{D}_\Phi$ and $\mathscr{S}_\Phi$ are convex subsets of $[0,1]^k$. (see [a] and [b])

[a] D. Mundici, Bookmaking on infinite-valued events. Int. J. Approx. Reasoning, 43(3): 223–249, 2006.

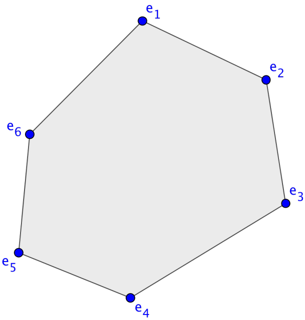
[b] T. Flaminio, Three Characterizations of Strict Coherence on Infinite-valued Events. Submitted, 2018

Theorem

Let Φ be a finite subset of $\mathcal{F}_n$. Then $\mathscr{D}_\Phi$ is a polytope and

$$\mathscr{S}_\Phi = \text{relint}\ \mathscr{D}_\Phi.$$[a]

[a] T. Flaminio, Three Characterizations of Strict Coherence on Infinite-valued Events. Submitted, 2018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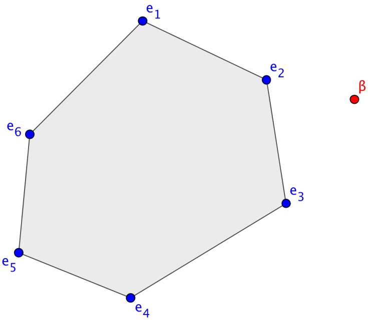
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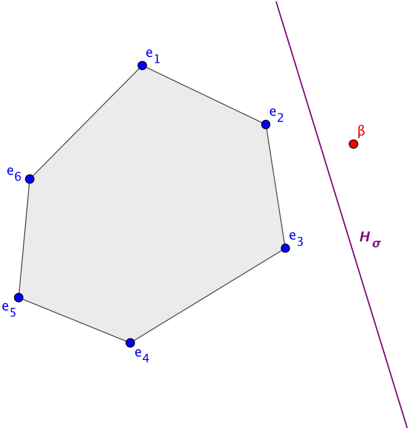
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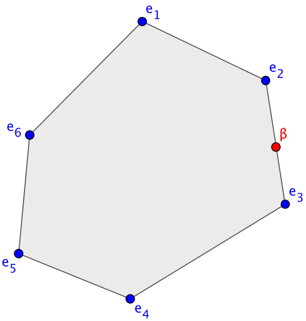
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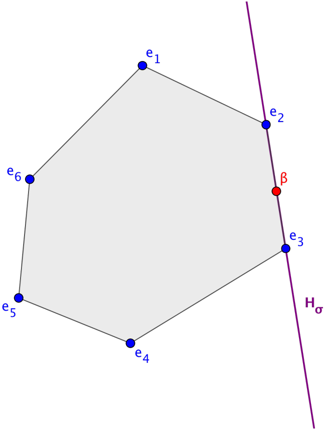
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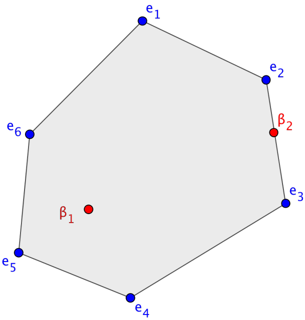
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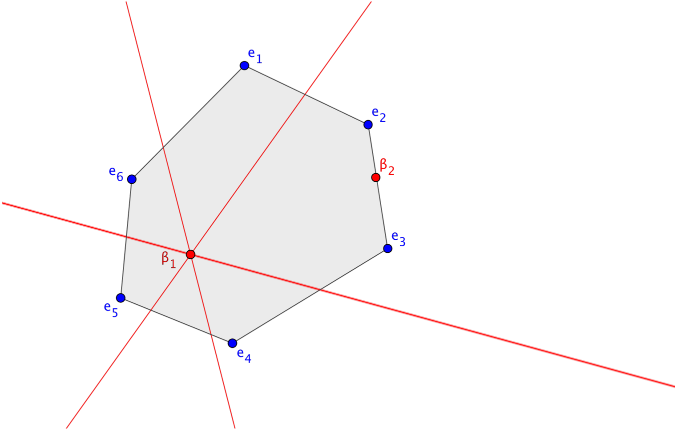
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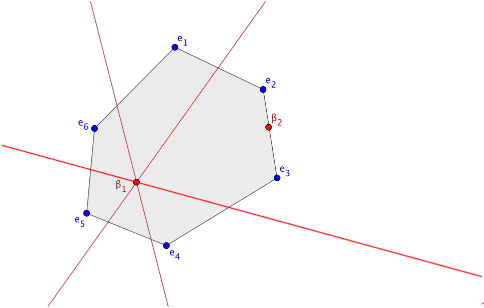
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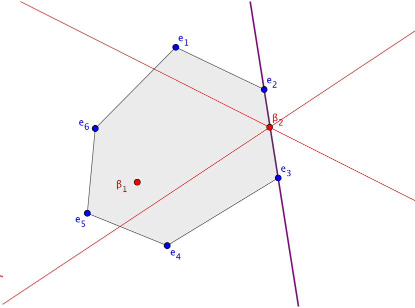
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The logic of coherence (from sets to formulas)

For each rational polyhedron P of $[0,1]^k$, there exists a formula Π_P of Łukasiewicz logic with k variables such that

$$P = \mathcal{M}od(\Pi_P).$$

Furthermore, if P_1 and P_2 are rational polyhedra,

$$P_1 \subseteq P_2 \text{ iff } \mathcal{M}od(\Pi_{P_1}) \subseteq \mathcal{M}od(\Pi_{P_2}) \text{ iff } \Pi_{P_1} \models \Pi_{P_2}.$$

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In general, the formula Π_P is not unique and its existence can be proved non constructively by employing the axiom of choice.

The previous result, due to Mundici, has been further extended by Marra and Spada who provided a categorical duality between the category of *finitely presented MV-algebras* (with homomorphisms) and the category of *rational polyhedra* (with $\mathbb{Z}$-maps).

However, it is not difficult to show that, for every rational polyhedra P, the formula Π_P can be effectively determined.

There exists an effective procedure Π which computes, for each finite $\Phi \subseteq \mathcal{F}_n$ with $|\Phi| = k$ and for each $\beta \in [0,1]^k$, Łukasiewicz formulas

$$\Pi_\Phi, \Pi_{(\mathrm{rb}\ \Phi)} \text{ and } \Pi_\beta$$

The onesets of which are the rational polyhedra $\mathscr{C}_\Phi$, rb $\mathscr{C}_\Phi$ and $\{\beta\}$.

Theorem (Flaminio, 2018)

Let $\Phi = f_1, \ldots, f_k$ be a finite set of $\mathcal{F}_n$ and let β be a rational-valued book on Φ. Then the following hold:

1. β is coherent iff $\vdash \Pi_\beta \to \Pi_\Phi$.
2. β is strictly coherent iff $\vdash \Pi_\beta \to \Pi_\Phi$ and $\not\vdash \Pi_\beta \to \Pi_{(\text{rb } \Phi)}$.

Thank you.